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LETTERS TO THE EDITOR

ON THE EIGENVALUES OF A VISCOUSLY DAMPED CANTILEVER CARRYING A TIP MASS

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1. INTRODUCTION

Although there is a vast number of papers in the vibration literature on the free vibrations of Bernoulli–Euler beams subjected to various boundary conditions, there are not many investigations (to the knowledge of the authors) on the eigencharacteristics of Bernoulli–Euler beams, viscously damped by absolute dampers. In reference [1], an approximate characteristic equation is set up for the free vibrations of Bernoulli–Euler beams restrained by numerous torsional and linear springs, carrying point and heavy masses and damped viscously by absolute dampers. The study in reference [2] is concerned with the sensitivity analysis of the eigenvalues of a viscously damped clamped–free beam where the analysis is also based on an approximate characteristic equation. The system considered in reference [3] is more general than in the previous study due to the inclusion of a tip mass. The present note is in some sense an extension of reference [3] because it is aimed here to derive the "exact" characteristic equation of the mechanical system investigated there.

2. THEORY

The system to be dealt with in the present study is shown in Figure 1. It is the same mechanical system as in reference [3]. The clamped–free beam carrying a tip mass M is damped at $x = \eta L$ by a viscous damper of constant c. Bending rigidity and mass per unit length of the beam are EI and m, respectively. The partial differential equation of the free bending vibrations of a uniform beam, according to Bernoulli–Euler theory is the well known equation

$$EIw^{IV}(x, t) + m\ddot{w}(x, t) = 0, \tag{1}$$

where $w(x, t)$ represents the bending displacement of the beam at point x and t. The primes and overdots denote partial derivatives with respect to x and t, respectively. The regions to the left and right of the damper are denoted as ① and ②, where $w_1(x, t)$ and $w_2(x, t)$ represent the bending displacements at the corresponding regions. The boundary and matching conditions are

$$w_1(0, t) = 0, \qquad w_1'(0, t) = 0, \qquad w_1(\eta L, t) = w_2(\eta L, t), \qquad w_2''(L, t) = 0,$$

$$w_1'(\eta L, t) = w_2'(\eta L, t), \qquad EIw_2'''(L, t) - M\ddot{w}_2(L, t) = 0, \qquad w_1''(\eta L, t) = w_2''(\eta L, t),$$

$$EIw_1'''(\eta L, t) - EIw_2'''(\eta L, t) - c\dot{w}_1(\eta L, t) = 0 \tag{2}$$

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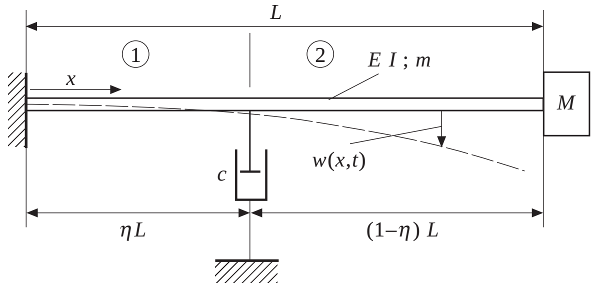

Figure 1. A viscously damped cantilever carrying a tip mass.

One assumes the solutions to be of the form

$$w_j(x, t) = W_j(x)\,\mathrm{e}^{\lambda t}, \qquad (j = 1, 2), \tag{3}$$

where λ denotes the unknown characteristic value of the system which is a complex number in general. In the expression above, both $w_j(x, t)$ and $W_j(x)$ represent complex valued functions. The essential point here is to imagine the actual bending displacements $w_j(x, t)$ as the real parts of some complex valued functions, for which the same notation is used for the sake of briefness. By putting the expressions (3) into the partial differential equation (1), the following ordinary differential equations for the functions $W_j(x)$ are obtained

$$W_j^{IV}(x) - \beta^4 W_j(x) = 0, \qquad (j = 1, 2), \tag{4}$$

where

$$\beta^4 = -m\lambda^2/EI \tag{5}$$

is introduced.

The general solutions of the differential equations (4) are

$$W_1(x) = C_1\,\mathrm{e}^{\beta x} + C_2\,\mathrm{e}^{-\beta x} + C_3\,\mathrm{e}^{\mathrm{i}\beta x} + C_4\,\mathrm{e}^{-\mathrm{i}\beta x},$$

$$W_2(x) = C_5\,\mathrm{e}^{\beta x} + C_6\,\mathrm{e}^{-\beta x} + C_7\,\mathrm{e}^{\mathrm{i}\beta x} + C_8\,\mathrm{e}^{-\mathrm{i}\beta x} \tag{6}$$

where C_1–C_8 represent eight integration constants yet to be determined and $\mathrm{i} = \sqrt{-1}$. Substitution of the expressions in (6) into the boundary and matching conditions (2) yields a set of linear, homogeneous equations consisting of eight equations for the determination of these constants. A non-trivial solution of the set is possible only if the determinant of the coefficients vanishes:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & \mathrm{i} & -\mathrm{i} & 0 & 0 & 0 & 0 \\ \mathrm{e}^{\eta\bar{\beta}} & \mathrm{e}^{-\eta\bar{\beta}} & \mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & \mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} & -\mathrm{e}^{\eta\bar{\beta}} & -\mathrm{e}^{-\eta\bar{\beta}} & -\mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & -\mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} \\ \mathrm{e}^{\eta\bar{\beta}} & -\mathrm{e}^{-\eta\bar{\beta}} & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} & -\mathrm{e}^{\eta\bar{\beta}} & \mathrm{e}^{-\eta\bar{\beta}} & -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} \\ \mathrm{e}^{\eta\bar{\beta}} & \mathrm{e}^{-\eta\bar{\beta}} & -\mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & -\mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} & -\mathrm{e}^{\eta\bar{\beta}} & -\mathrm{e}^{-\eta\bar{\beta}} & \mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & \mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} \\ (1-a)\,\mathrm{e}^{\eta\bar{\beta}} & -(1+a)\,\mathrm{e}^{-\eta\bar{\beta}} & -(\mathrm{i}+a)\,\mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & (\mathrm{i}-a)\,\mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} & -\mathrm{e}^{\eta\bar{\beta}} & \mathrm{e}^{-\eta\bar{\beta}} & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} \\ 0 & 0 & 0 & 0 & \mathrm{e}^{\bar{\beta}} & \mathrm{e}^{-\bar{\beta}} & -\mathrm{e}^{\mathrm{i}\bar{\beta}} & -\mathrm{e}^{-\mathrm{i}\bar{\beta}} \\ 0 & 0 & 0 & 0 & (1-b)\,\mathrm{e}^{\bar{\beta}} & -(1+b)\,\mathrm{e}^{-\bar{\beta}} & -(\mathrm{i}+b)\,\mathrm{e}^{\mathrm{i}\bar{\beta}} & (\mathrm{i}-b)\,\mathrm{e}^{-\mathrm{i}\bar{\beta}} \end{vmatrix} = 0. \tag{7}$$

By applying elementary row and column operations, this equation can be brought into a simpler form:

$$\begin{vmatrix} \mathrm{e}^{-\eta\bar{\beta}} & \mathrm{e}^{\eta\bar{\beta}} & \mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} & \mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & 0 & 0 & 0 & 0 \\ \mathrm{e}^{-\eta\bar{\beta}} & -\mathrm{e}^{\eta\bar{\beta}} & \mathrm{i}\mathrm{e}^{-\mathrm{i}\eta\bar{\beta}} & -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\eta\bar{\beta}} & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & \mathrm{i} & -\mathrm{i} & -1 & 1 & -\mathrm{i} & \mathrm{i} \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1-a & -(1+a) & -(\mathrm{i}+a) & \mathrm{i}-a & -1 & 1 & \mathrm{i} & -\mathrm{i} \\ 0 & 0 & 0 & 0 & \mathrm{e}^{(1-\eta)\bar{\beta}} & \mathrm{e}^{-(1-\eta)\bar{\beta}} & -\mathrm{e}^{\mathrm{i}(1-\eta)\bar{\beta}} & -\mathrm{e}^{-\mathrm{i}(1-\eta)\bar{\beta}} \\ 0 & 0 & 0 & 0 & (1-b)\,\mathrm{e}^{(1-\eta)\bar{\beta}} & -(1+b)\,\mathrm{e}^{-(1-\eta)\bar{\beta}} & -(\mathrm{i}+b)\,\mathrm{e}^{\mathrm{i}(1-\eta)\bar{\beta}} & (\mathrm{i}-b)\,\mathrm{e}^{-\mathrm{i}(1-\eta)\bar{\beta}} \end{vmatrix} = 0. \tag{8}$$

where the following abbreviations have been introduced

$$\bar{\beta} = \beta L, \qquad \beta_M = M/mL, \qquad b = -\beta_M \bar{\beta}, \qquad d = c/mL\omega_0, \qquad \omega_0^2 = EI/mL^4,$$

$$a = \pm \mathrm{i} d/\bar{\beta}. \tag{9}$$

It is seen from the above that the parameters a and b are related to the effect of viscous damping and the tip mass, respectively.

The determinantal equation (8) is in some sense the characteristic equation of the system in Figure 1. The solution of this equation with respect to $\bar{\beta}$ yields via

$$\lambda = \pm \mathrm{i}\omega_0 \bar{\beta}^2 \tag{10}$$

the unknown complex eigenvalues λ of the mechanical system.

It is in order, for comparison of the numerical values in the next section, to give here another expression for the approximate determination of the eigenvalues of the same system, taken from reference [3]. There, in the context of the sensitivity analysis of the system under investigation, the following $2n \times 2n$ matrix was introduced

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \tag{11}$$

In addition to those given (9), the following abbreviations are defined:

$$\lambda^* = \lambda/\omega_0, \qquad \mathbf{a}(1) = [a_1(1), \ldots, a_k(1), \ldots, a_n(1)]^{\mathrm{T}},$$

$$a_k(1) = \cosh \bar{\beta}_k^* - \cos \bar{\beta}_k^* - \bar{\eta}_k (\sinh \bar{\beta}_k^* - \sin \bar{\beta}_k^*),$$

$$\bar{\eta}_k = (\cosh \bar{\beta}_k^* + \cos \bar{\beta}_k^*)/(\sinh \bar{\beta}_k^* + \sin \bar{\beta}_k^*)$$

$$\bar{\beta}_1^* = 1{\cdot}875104, \qquad \bar{\beta}_2^* = 4{\cdot}694091, \qquad \bar{\beta}_3^* = 7{\cdot}854757, \ldots,$$

$$\mathbf{a}(\eta) = [a_1(\eta), \ldots, a_k(\eta), \ldots, a_n(\eta)]^{\mathrm{T}},$$

$$a_k(\eta) = \cosh \bar{\beta}_k^* \eta - \cos \bar{\beta}_k^* \eta - \bar{\eta}_k (\sinh \bar{\beta}_k^* \eta - \sin \bar{\beta}_k^* \eta),$$

$$\mathbf{K} = \mathbf{diag}(\bar{\beta}_k^{*4} \eta), \qquad \mathbf{I} = n \times n (\text{identity matrix}), \qquad \mathbf{M} = \mathbf{I} + \beta_M \mathbf{a}(1)\mathbf{a}^{\mathrm{T}}(1),$$

$$\mathbf{D} = d\mathbf{a}(\eta)\mathbf{a}^{\mathrm{T}}(\eta). \tag{12}$$

The eigenvalues of the above matrix **A** yield good approximate values for the eigenvalues of the mechanical system in Figure 1.

3. NUMERICAL APPLICATIONS

This section is devoted to the testing of the reliability of the analytical expressions obtained. Firstly, the special case $c = 0$, where damping is not present, will be considered.

TABLE 1

Effect of the variation of the tip mass ratio β_M on the dimensionless frequency parameter $\overline{\beta}$.

$\overline{\beta}$	β_M					
	0	0·20	0·40	0·60	0·80	1·00
$\overline{\beta}_1$	1·87510407	1·61639966	1·47240849	1·37566854	1·30408675	1·24791741
$\overline{\beta}_2$	4·69409113	4·26706157	4·14443036	4·08665324	4·05307815	4·03113944
$\overline{\beta}_3$	7·85475744	7·31837267	7·21548589	7·17252465	7·14898484	7·13413224
$\overline{\beta}_4$	10·99554073	10·40156263	10·31780693	10·28498044	10·26748665	10·25662107
$\overline{\beta}_5$	14·13716839	13·50670225	13·43667566	13·41020846	13·39631447	13·38775633

The first five roots of the characteristic equation (8) obtained with the help of MATHEMATICA are given in Table 1 with respect to the tip mass parameter β_M. The results shown are exactly the same as in reference [4]. For the damped system, the physical data of the systems in [2, 3] are taken. $E = 7 \times 10^{10}$ N/m^2, $I = (0{\cdot}05 \times 0{\cdot}005^3)/12$ m^4, $L = 1$ m, $mL = 0{\cdot}675$ kg, $\eta = 0{\cdot}2$, $c = 5$ N/(m/s), $\beta_M = 3$. The last value means that the tip mass is 3 times the beam mass.

The first five pairs of eigenvalues λ of the system (arranged with respect to the magnitude of the imaginary parts) are given in Table 2. The complex numbers in the first column represent the "exact" eigenvalues λ, which are obtained from the solution of equation (8) with MATHEMATICA with respect to $\overline{\beta}$ and then using (10). The complex numbers in the second column are obtained from the eigenvalues λ^* of the matrix **A** in (11) and then by multiplying them by ω_0. The eigenvalues λ^* are calculated with the help of MATLAB, by taking $n = 20$ in (11, 12). The comparison of the complex numbers in both columns of Table 2 reveals that the eigenvalues of the matrix **A** yield quite good approximations when compared to the "exact" eigenvalues of the mechanical system in Figure 1. The magnitudes of both the real and imaginary parts of the approximate eigenvalues are slightly greater than those of the "exact" eigenvalues, as expected.

It is quite instructive to report also on the experience gained during the solution of the complex equation (8) with respect to $\overline{\beta}$. As seen from the definitions in (9), the parameter a may have positive or negative signs. In case of the (+) sign, if some $\overline{\beta} = \alpha + \beta\mathrm{i}$ is a root of equation (8) then $-\alpha - \mathrm{i}\beta$, $\beta + \mathrm{i}\alpha$ and $-\beta - \mathrm{i}\alpha$ are roots also. In case of (−) sign, $-\alpha + \mathrm{i}\beta$, $\alpha - \mathrm{i}\beta$, $\beta - i\alpha$ and $-\beta + \mathrm{i}\alpha$ also represent roots of the equation. However, all of these roots yield one pair of complex conjugate number $\lambda_{1,2}$ which are physically meaningful (i.e., negative real parts), when the same signs are selected in both (9) and (10). Hence, it is sufficient to consider only the positive sign in the definitions of a and λ.

TABLE 2

Eigenvalues λ of the system. First column: from the roots of the characteristic equation (8), *via* (10); *second column: eigenvalues of matrix* A *multiplied by ω_0*

from eq. (8)	from matrix **A**
$-3{\cdot}660439\ 10^{-3} \pm \mathrm{i}\ 7{\cdot}076019$	$-3{\cdot}660459\ 10^{-3} \pm \mathrm{i}\ 7{\cdot}076091$
$-0{\cdot}785780 \pm \mathrm{i}\ 115{\cdot}536606$	$-0{\cdot}786144 \pm \mathrm{i}\ 115{\cdot}54901$
$-4{\cdot}297368 \pm \mathrm{i}\ 369{\cdot}632796$	$-4{\cdot}300372 \pm \mathrm{i}\ 369{\cdot}77255$
$-8{\cdot}057146 \pm \mathrm{i}\ 768{\cdot}481528$	$-8{\cdot}075455 \pm \mathrm{i}\ 769{\cdot}09340$
$-7{\cdot}385813 \pm \mathrm{i}\ 1312{\cdot}397925$	$-7{\cdot}378600 \pm \mathrm{i}\ 1314{\cdot}2003$

4. CONCLUSIONS

The present study is concerned with the investigation of the eigencharacteristics of a special system consisting of a viscously damped, clamped–free Bernoulli–Euler beam carrying a tip mass. The exact characteristic equation is established via a boundary value problem formulation. Numerical results obtained by solving this complex equation, are given in the form of Tables which indicate the reliability of the derived analytical expressions.

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